

## SENSITIVITY ANALYSIS FOR LARGE-SCALE PROBLEMS

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## INTRODUCTION

Due to the wide variety of uses of sensitivity derivatives, the development of efficient computational procedures for the calculation of these derivatives has attracted considerable attention in recent years. The calculation of sensitivity derivatives forms the backbone of many optimization procedures and is the major contributor to the cost and time of optimization of large systems. In addition, sensitivity derivatives have several other applications in structural mechanics including approximate analysis (and reanalysis) techniques, analytical model improvement, and assessment of design trends. A review of the state of the art in sensitivity calculations is contained in a survey paper (Ref. 1), a monograph (Ref. 2), as well as in some papers in these proceedings. Despite all the recent advances made, the calculation of sensitivity derivatives for large structural systems (with large number of degrees of freedom and design variables), is quite expensive even on present-day large computers.

The present study focuses on the development of efficient techniques for calculating sensitivity derivatives. Specifically, the objective and scope of the present paper are listed in Fig. 1. The objective is to present a computational procedure for calculating sensitivity derivatives as part of performing structural reanalysis for large-scale problems. The scope of the paper is limited to framed type structures. Both linear static analysis and free-vibration eigenvalue problems are considered.

Objective

- To present a computational procedure for calculating sensitivity derivatives as part of performing structural reanalysis for large-scale problems

Scope

- Frame-type structures
- Linear static analysis
- Eigenvalue problems

Figure 1

## BASIC IDEA AND KEY ELEMENTS OF THE PROPOSED PROCEDURE

The basic idea and the three key elements of the proposed procedure are listed in Fig. 2. The basic idea is to generate the solution of the modified structure using large perturbations from that of the original structure. The three key elements are: a) lumping of the large number of design variables into one (or a small number of) tracing parameters; b) application of operator splitting/reduction technique; and c) for very large problems, use of single-level or multilevel substructuring. Only the first two key elements are discussed in this paper. The application of operator splitting/reduction technique proved to be effective in reducing the computational effort in a number of structural mechanics problems (see, for example, Refs. 3 to 6).

### Basic idea

- Solution for modified structure is obtained using large perturbations from that of original structure

### Key elements

- Lumping of design variables into tracing parameter(s)
- Application of operator splitting/reduction technique
- Use of multilevel substructuring (for very large problems)

Figure 2

## APPLICATION TO LINEAR STATIC ANALYSIS

In Fig. 3 the application of the proposed procedure to linear static analysis is outlined. The governing finite element equations of the original and modified structures are shown. The global stiffness matrices, load vectors, and responses of the original and modified structures are designated by  $[K]_0$ ,  $[K]$ ;  $\{P\}_0$ ,  $\{P\}$ ; and  $\{X\}_0$ ,  $\{X\}$ , respectively. The original and modified structure characteristics correspond to the values of  $d_i^0$  and  $d_i^*$  of the design variables, respectively.

The operator splitting technique is now applied, and the equations of the modified structure are expressed in terms of the original structure equations plus correction terms. A tracing parameter  $\lambda$  is introduced and is attached to the correction terms. The tracing parameter is dimensionless and identifies *all* the design modifications. The original structure equations correspond to  $\lambda=0$ , and the modified structure equations correspond to  $\lambda=1$ .

### Operator splitting

#### Original structure

$$[K]_0 \{X\}_0 = \{P\}_0 ; [K]_0 = [K(d_i^0)]_0$$

#### Modified structure

$$[K] \{X\} = \{P\} ; [K] = [K(d_i^*)]$$

$$\text{or } \left[ [K]_0 + \lambda ([K] - [K]_0) \right] \{X\} = \{P\}_0 + \lambda (\{P\} - \{P\}_0)$$

$$\lambda = 0 \rightarrow \text{Original structure}$$

$$\lambda = 1 \rightarrow \text{Modified structure}$$

Figure 3

The response of the modified structure,  $\{X\}$ , is now expressed as a linear combination of a few preselected global approximation vectors (or modes). This is expressed by the transformation shown in Fig. 4. The columns of the matrix  $[\Gamma]$  are the global approximation vectors, and the elements of the vector  $\{\psi\}$  are the amplitudes of the approximation vectors which are, as yet, unknowns. Note that the number of global approximation vectors,  $r$ , is considerably smaller than the total number of degrees of freedom,  $n$ .

A Rayleigh-Ritz technique is now used to approximate the governing equations of the modified structure by a much smaller system of equations in the unknowns  $\{\psi\}$ .

### Basis reduction

$$\{X\}_{n,1} = [\Gamma]_{n,r} \{\psi\}_{r,1} \quad ; \quad r \ll n$$

where  $\{\psi\}$  = amplitudes of global approximation vectors

### Reduced system of equations

Rayleigh -Ritz technique used to approximate the equations of the modified structure

$$\left[ [\tilde{K}]_0 + \lambda \left( [\tilde{K}] - [\tilde{K}]_0 \right) \right] \{\psi\} = \{\tilde{P}\}_0 - \lambda \left( \{\tilde{P}\} - \{\tilde{P}\}_0 \right)$$

$$\text{where } [\tilde{K}]_0 = [\Gamma]^t [K]_0 [\Gamma]$$

$$[\tilde{K}] = [\Gamma]^t [K] [\Gamma]$$

$$\{\tilde{P}\}_0 = [\Gamma]^t \{P\}_0$$

$$\{\tilde{P}\} = [\Gamma]^t \{P\}$$

Figure 4

# SELECTION AND EVALUATION OF THE GLOBAL APPROXIMATION

The effectiveness of the proposed procedure depends, to a great extent, on the proper choice of the global approximation vectors. In the present study the global approximation vectors are selected to be the response of the original structure,  $\{X\}_0$ , and its various-order derivatives with respect to the parameter  $\lambda$ . The recursion relations for evaluating the approximation vectors are obtained by successive differentiation of the original finite element equations. Note that the matrix on the left hand sides of these equations,  $[K]_0$ , is the same (see Fig. 5).

$$\{X\} = [\Gamma] \{\psi\}$$

$$\text{where } [\Gamma] = \left[ \{X\}_0 \quad \left\{ \frac{\partial X}{\partial \lambda} \right\}_0 \quad \left\{ \frac{\partial^2 X}{\partial \lambda^2} \right\}_0 \dots \right]$$

$$[K]_0 \{X\}_0 = \{P\}_0$$

$$[K]_0 \left\{ \frac{\partial X}{\partial \lambda} \right\}_0 = - \left( [K] - [K]_0 \right) \{X\}_0 + \{P\} - \{P\}_0$$

$$[K]_0 \left\{ \frac{\partial^2 X}{\partial \lambda^2} \right\}_0 = -2 \left( [K] - [K]_0 \right) \left\{ \frac{\partial X}{\partial \lambda} \right\}_0$$

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Same left hand side.

Figure 5

## COMPUTATIONAL PROCEDURE

The computational procedure for generating the solution of the modified structure and the sensitivity derivatives is outlined in Fig. 6.

The first step is to generate the global approximation vectors at  $\lambda=0$  through decomposition of the stiffness matrix of the original structure. The derivatives with respect to  $\lambda$  provide information about the sensitivity of the response to *all* the design modifications. Because  $\lambda$  is dimensionless, the derivatives with respect to  $\lambda$  have the same dimensions as the original response quantities, and consequently an assessment of the effect of design modifications on the response can be easily made.

The second step is to generate the reduced equations and solve them for the amplitudes of the global approximation vectors.

- Evaluate global approximation vectors at  $\lambda = 0$
- Derivatives with respect to  $\lambda$  represent sensitivity of the response to design modifications
- Generate reduced equations
- Solve reduced equations and find amplitudes of global approximation vectors

Figure 6

# RELATIONSHIP BETWEEN THE PRECONDITIONED CONJUGATE GRADIENT (PCG) TECHNIQUE AND THE PROPOSED COMPUTATIONAL PROCEDURE

If the proposed computational procedure is contrasted with the preconditioned conjugate gradient (PCG) technique in which the preconditioning matrix is selected to be the global stiffness matrix of the original structure,  $[K]_0$ , the relationships shown in Fig. 7 can be identified. These relations express the preconditioned residuals  $\{y\}_0, \{y\}_1, \dots$  of the PCG technique in terms of the global approximation vectors of the foregoing technique,  $\{\frac{\partial X}{\partial \lambda}\}_0, \{\frac{\partial^2 X}{\partial \lambda^2}\}_0, \dots$

## Equivalence

$$([K]_0 + \lambda([K] - [K]_0))\{X\} = \{P\}_0 + \lambda(\{P\} - \{P\}_0)$$

$[K]_0$  = preconditioning matrix

PCG	Proposed procedure
$\{X\}_0$	$\{X\}_0$
<u>Preconditioned residuals</u>	
$\{y\}_0$	$\{\frac{\partial X}{\partial \lambda}\}_0$
$\{y\}_1$	$(1 - c_0) \{\frac{\partial X}{\partial \lambda}\}_0 + \frac{c_0}{2} \{\frac{\partial^2 X}{\partial \lambda^2}\}_0$
$\{y\}_i$	$\sum_{j=1}^{i+1} c_j \{\frac{\partial^j X}{\partial \lambda^j}\}_0$

Figure 7

## IMPLICATIONS OF THE SIMILARITIES BETWEEN THE PROPOSED PROCEDURE AND THE PCG TECHNIQUE

The implications of the similarities between the proposed computational procedure and the PCG technique are listed in Fig. 8.

For the PCG technique, the similarities can be exploited to provide a rational approach for selecting the preconditioning matrix (as the global stiffness matrix of the original structure), and a physical meaning for the preconditioned residual vectors (in terms of sensitivity derivatives).

For the proposed procedure, some of the work done on parallelizing the PCG on multiprocessor computers can be exploited.

### PCG

- Rational choice for preconditioning matrix
- Physical meaning for preconditioned residuals  
(in terms of sensitivity derivatives)

### Proposed procedure

- Exploiting work done on parallelizing PCG on  
multiprocessor computers

Figure 8



## APPLICATION TO EIGENVALUE PROBLEMS

The application of the proposed computational procedure to free vibration (eigenvalue) problems is outlined in Fig. 9. The governing equations of the original and modified structures are shown. Again, the operator splitting technique is applied, and the stiffness and mass matrices of the modified structure are written as the sum of the corresponding matrices of the original structure plus correction terms. The correction terms are identified by the tracing parameter  $\lambda$ . When  $\lambda=0$ , the original structure equations are recovered, and when  $\lambda=1$  the modified structure equations are obtained.

### Operator splitting

#### Original structure

$$\left[ [K]_0 - \Omega [M]_0 \right] \{X\}_0 = 0$$

#### Modified structure

$$\left[ [K] - \Omega [M] \right] \{X\} = 0$$

or,

$$\left[ \left( [K]_0 - \Omega [M]_0 \right) + \lambda \left( [K]_a - \Omega [M]_a \right) \right] \{X\} = 0$$

$$\lambda = 0 \longrightarrow \text{original structure}$$

$$\lambda = 1 \longrightarrow \text{modified structure}$$

where  $[K]_a = [K] - [K]_0$

$$[M]_a = [M] - [M]_0$$

Figure 9

The application of the reduction method to the free vibration problem is outlined in Fig. 10. As in the static analysis, the eigenvectors of the modified structure,  $\{X\}$ , are approximated by a linear combination of a few global approximation vectors. This is accomplished by the transformation shown. An efficient choice of the approximation vectors was found to be a few eigenvectors for the original structure (corresponding to  $\lambda=0$ ) and their derivatives with respect to  $\lambda$ , evaluated at  $\lambda=0$ .

Then, the Rayleigh-Ritz technique is used to approximate the original large eigenvalue problem by the reduced one shown in Fig. 10. The solution of the reduced eigenvalue problem gives the amplitudes of the global approximation vectors.

### Basis reduction

- Eigenvectors of modified structure,  $\{X\}$ , are approximated by:

$$\{X\}_{n,1} = [\Gamma]_{n,r} \{\psi\}_{r,1} \quad ; \quad r \ll n$$

where

$$[\Gamma] = \left[ \{X\}_1 \left\{ \frac{\partial X}{\partial \lambda} \right\}_1 \left\{ \frac{\partial^2 X}{\partial \lambda^2} \right\}_1 \cdots \{X\}_2 \left\{ \frac{\partial X}{\partial \lambda} \right\}_2 \left\{ \frac{\partial^2 X}{\partial \lambda^2} \right\}_2 \cdots \right]_{\lambda=0}$$

### Reduced system of equations

- Rayleigh-Ritz technique is used to approximate the original eigenvalue problem by a reduced one

$$\left[ \left( [\tilde{K}]_0 - \Omega [\tilde{M}]_0 \right) + \lambda \left( [\tilde{K}]_a - \Omega [\tilde{M}]_a \right) \right] \{\psi\} = 0$$

where

$$[\tilde{K}]_0 = [\Gamma]^t [K]_0 [\Gamma]$$

$$[\tilde{K}]_a = [\Gamma]^t [K]_a [\Gamma]$$

$$[\tilde{M}]_0 = [\Gamma]^t [M]_0 [\Gamma]$$

$$[\tilde{M}]_a = [\Gamma]^t [M]_a [\Gamma]$$

Figure 10

## EVALUATION OF GLOBAL APPROXIMATION VECTORS

The recursion equations used in generating the eigenvectors at  $\lambda=0$  and their first derivatives with respect to  $\lambda$  are shown in Fig. 11. Note that the left hand sides of all these equations are the same. The expression of the first derivatives of the eigenvalues with respect to  $\lambda$  appearing on the right hand sides of the equations are given in Fig. 11.

Since the matrix on the left hand side of the recursion equations used in evaluating these derivatives is singular, the solution of each set of equations can be expressed as the sum of a homogeneous solution (multiple of the eigenvector) and a particular solution,  $\{Q\}_1$ . The equations used in evaluating the particular solution  $\{Q\}_1$  are given in Fig. 11. The details of this procedure are given in Ref. 5.

### Recursion formulas

$$\begin{aligned}
 & \left[ [K]_0 - \Omega [M]_0 \right] \{X\} = 0 \\
 & \left[ [K]_0 - \Omega [M]_0 \right] \left\{ \frac{\partial X}{\partial \lambda} \right\} = \frac{\partial \Omega}{\partial \lambda} \left[ [M]_0 + \lambda [M]_a \right] \{X\} - \left[ [K]_a - \Omega [M]_a \right] \{X\} \\
 & \quad \vdots \\
 & \quad \vdots \\
 & \frac{\partial \Omega}{\partial \lambda} = \{X\}^t \left[ [K]_a - \Omega [M]_a \right] \{X\} / \{X\}^t \left[ [M]_0 + \lambda [M]_a \right] \{X\} \\
 & \quad \vdots \\
 & \quad \vdots \\
 & \left\{ \frac{\partial X}{\partial \lambda} \right\} = \{Q\}_1 + C_1 \{X\} \\
 & \quad \vdots \\
 & \quad \vdots
 \end{aligned}$$

Where  $\{Q\}_1$  = particular solution.

Figure 11

## COMPUTATIONAL PROCEDURE FOR EIGENVALUE PROBLEMS

The procedure for extracting the eigenvectors of the modified structure and for generating the sensitivity of the eigenvectors to design modifications is outlined in Fig. 12.

First: A few eigenvectors of the original structure (corresponding to  $\lambda=0$ ) are generated.

Second: The derivatives of the eigenvectors with respect to  $\lambda$  are generated at  $\lambda=0$ . In the process, derivatives of the eigenvalues are also computed. These derivatives provide sensitivity information regarding the effect of all the design modifications on the eigenvectors and eigenvalues. The reduced equations are generated.

Third: The reduced eigenvalue problem is solved at  $\lambda=1$ .

- Generate eigenvectors for original structure ( $\lambda = 0$ )
- Generate global approximation vectors (derivatives of eigenvectors w.r.t.  $\lambda$ ) and reduced equations. In the process, derivatives of  $\Omega$  w.r.t.  $\lambda$  are computed
- Solve reduced eigenvalue problem at  $\lambda = 1$

Figure 12

# CANTILEVERED LATTICE TRUSS

To assess the effectiveness of the proposed computational procedure, a number of problems were solved by this procedure. Comparison was made with the direct solution of the structure. Herein a typical problem of a five-bay cantilevered lattice truss is considered (see Fig. 13). In the original structure all the longerons had the same cross section, and all the battens and diagonals had the same cross section. The design variables consisted of the cross sectional areas, moments of inertia and torsional constants. The characteristics of the original and modified structures are given in Fig. 13.

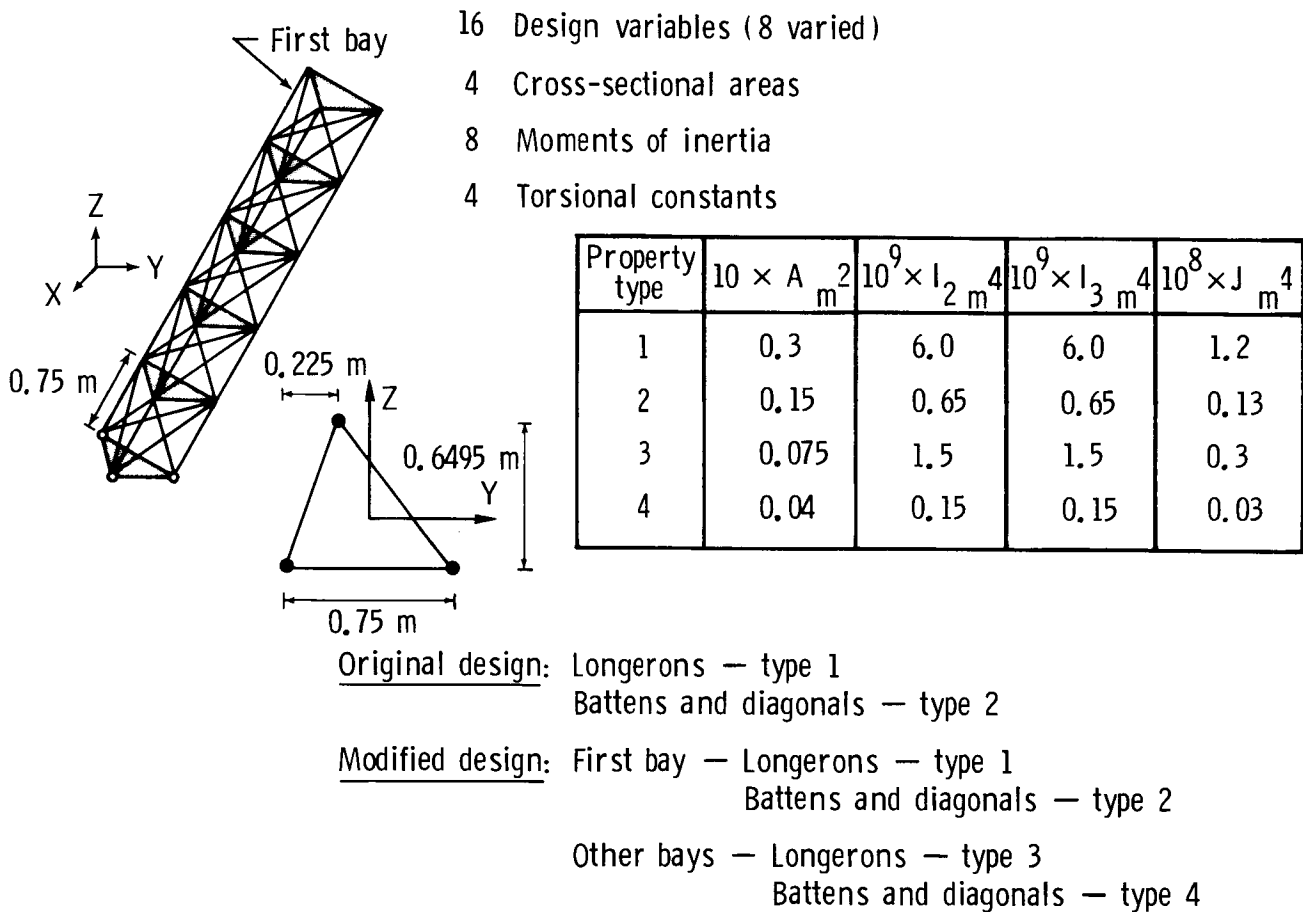


Figure 13

## STATIC LOADING

The first problem considered is that of the static response due to a transverse load in the  $z$  direction at the free end of the cantilever. Figure 14 shows a summary of the results. The transverse displacement  $w$  and rotation at the free end (point  $a$ ) of the original structure are given. The sensitivity of these quantities to design modifications is provided by  $\left\{\frac{\partial X}{\partial \lambda}\right\}_0$  and  $\left\{\frac{\partial^2 X}{\partial \lambda^2}\right\}_0$ . Also shown in Fig. 14 are the corresponding  $w$  and  $\phi_2$  for the modified structure (which are considerably larger than those for the original structure). The solution obtained using the proposed procedure with four global approximation vectors was identical to the direct solution of the modified structure to at least three significant digits.

### Static analysis

			$w$ at $a$	$\phi_2$ at $a$
Original structure $\lambda = 0$	$\{X\}$		0.102	-0.0397
	$\left\{\frac{\partial X}{\partial \lambda}\right\}_0$		0.0737	-0.0257
	$\left\{\frac{\partial^2 X}{\partial \lambda^2}\right\}_0$		0.110	-0.0388
Modified structure $\lambda = 1$	$\{X\}$	$r = 2$	0.394	-0.142
		$r = 4$	0.394	-0.143
		Full system	0.394	-0.143

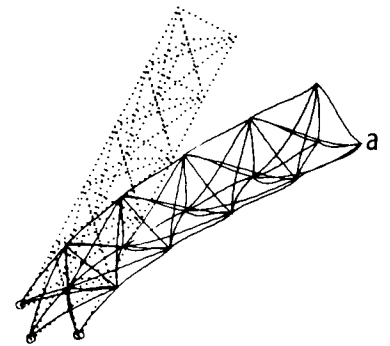


Figure 14

# FREE VIBRATIONS

The second problem considered is that of the free vibrations of the same lattice structure. Figure 15 shows a summary of the results. The first three eigenvalues (squares of the vibration frequencies) and their first two derivatives with respect to  $\lambda$  are listed. The corresponding eigenvalues of the modified structure are also listed. The frequencies predicted by the proposed procedure with eight approximation vectors (four eigenvectors and their first derivatives with respect to  $\lambda$ ) and twelve approximation vectors (four eigenvectors and their first two derivatives with respect to  $\lambda$ ) are listed. The predictions of the eight-vector approximation are accurate for the first two eigenvalues, but not the succeeding ones. On the other hand, the predictions of the twelve-vector approximation are accurate for the first three eigenvalues.

			Mode		
			1	2	3
Original structure $\lambda = 0$	$10^{-5} \times \Omega$		0.324	0.502	1.815
	$10^{-5} \times \frac{\partial \Omega}{\partial \lambda}$		-0.134	-0.207	-0.486
	$10^{-5} \times \frac{\partial^2 \Omega}{\partial \lambda^2}$		-0.0911	-0.141	-0.472
Modified structure $\lambda = 1$	$10^{-5} \times \Omega$	$r = 8$	0.122	0.191	0.583
		$r = 12$	0.122	0.191	0.896
		Full system	0.122	0.191	0.896

Figure 15

## MODE SHAPES

The first three mode shapes of the modified structure are shown in Fig. 16. Note that the first two vibration modes are bending modes and the third is a torsional mode.

Free vibrations

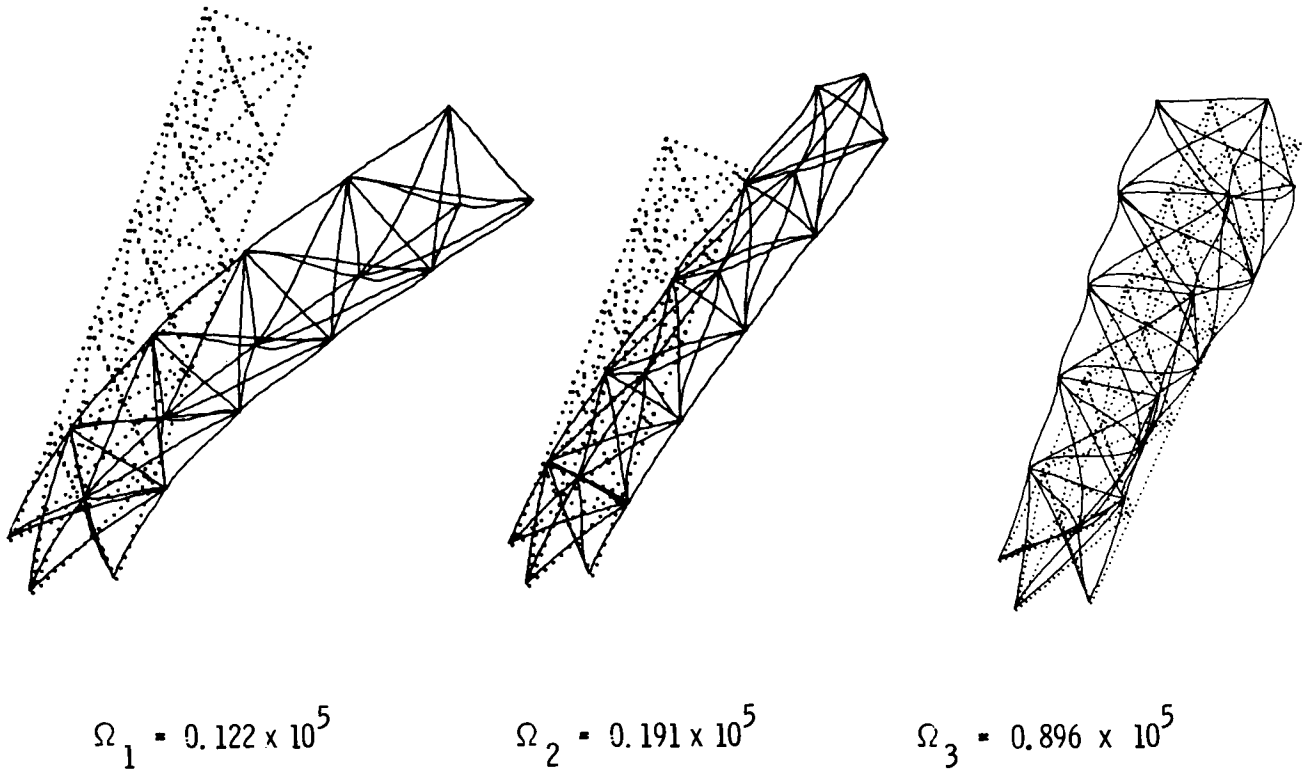


Figure 16



## SUMMARY

In summary, a computational procedure has been developed for calculating the sensitivity derivatives of large structural systems as part of structural reanalysis (see Fig. 17). The three key elements of the procedure are:

- a) lumping of the large number of design variables into one (or small number of) tracing parameter(s);
- b) application of operator splitting/reduction technique; and,
- c) for very large problems use of multilevel substructuring technique.

The proposed procedure can be considered as a general computational strategy for generating the response of the modified structure using large perturbations from the response of the original structure.

For static problems the similarities between the proposed procedure and preconditioned conjugate gradient technique are identified and are exploited to provide a rational procedure for selecting the preconditioning matrix and a physical meaning for the preconditioned residual vectors.

Future work includes:

- o extension to more complex structures and to shape design modifications
- o generation of sensitivity information with respect to design variables.

- Computational procedure presented for calculating sensitivity derivatives as part of performing structural reanalysis for large-scale problems
  - Lumping of design variables into tracing parameter(s)
  - Application of operator splitting/reduction technique
  - Use of multilevel substructuring
- Future work includes:
  - Extension to more complex structures and to shape design modifications
  - Generation of sensitivity information w.r.t. design variables

Figure 17

## REFERENCES

1. Adelman, H. M. and Haftka, R. T.: Sensitivity Analysis of Discrete Structural Systems. AIAA Journal, vol. 24, no. 5, May 1986, pp. 823-832.
2. Haug, E. J., Choi, K. K., and Komkov, V.: Design Sensitivity Analysis of Structural Systems, Academic Press, New York, 1986.
3. Noor, A. K. and Peters, J. M.: Model Size Reduction Technique for the Analysis of Symmetric Anisotropic Structures. Engineering Computations, vol. 2, no. 4, December 1985, pp. 285-292.
4. Noor, A. K. and Peters, J. M.: Nonlinear Analysis of Anisotropic Panels. AIAA Journal, vol. 24, no. 9, September 1986, pp. 1545-1553.
5. Noor, A. K. and Whitworth, S. L.: Model Size Reduction for the Buckling and Vibration Analyses of Anisotropic Panels. Journal of the Engineering Mechanics Division, ASCE, vol. 113, no. 2, February 1987, pp. 170-185.
6. Noor, A. K. and Whitworth, S. L.: Model Size Reduction for the Analysis of Symmetric Structures with Unsymmetric Boundary Conditions. International Journal for Numerical Methods in Engineering, 1987.